Introduction to random zeros of holomorphic sections
Part 2: Bergman kernel and equidistribution of random zeros

21/03/2024

§0 Revisit Standard Gaussian holomorphic Sections (from Lecture 1)

X connected complex mfd of $\dim = n$ (without boundary) J complex structure

ω Hermitian metric J-invariant

$g^{TX}(\cdot,\cdot) := \omega(\cdot, J\cdot)$ Riemannian metric

Riemannian volume form $dV = \frac{\omega^n}{n!}$

} smooth

(L, h_L) holomorphic line bundle on X ~~with~~ C^∞- Hermitian metric h_L.

$H^0_{(2)}(X,L) = H^0(X,L) \cap \mathcal{L}^2(X,L)$ Hilbert space.

$d = \dim_{\mathbb{C}} H^0_{(2)}(X,L) \in \mathbb{N} \cup \{\infty\}$

Bergman projector:

$$B: \mathcal{L}^2(X,L) \xrightarrow{\perp} H^0_{(2)}(X,L)$$

Bergman kernel $B(x,y)$ $X \times X$

$$(BS)(x) = \int_X B(x,y) S(y)\, dV(y)$$

Fix $\{S_j\}_{j=1}^d$ ONB of $H^0_{(2)}(X,L)$

$$B(x,y) = \sum_{j=1}^d S_j(x) \otimes S_j(y)^* \qquad C^\infty \text{ in } x, y$$

$$B(x) := B(x,x) = \sum_j |S_j(x)|^2_{h_L} \geq 0.$$

Def: Standard Gaussian hol. section

$$S(L) := \sum_{j=1}^d \eta_j S_j$$

where $\{\eta_j\}_j$ i.i.d. $\sim \mathcal{N}_{\mathbb{C}}(0,1)$

"a canonical Gaussian section constructed from $H^0_{(2)}(X,L)$"

$d \geq 1$ 流动形 current

By Thm 2 (Lecture 1)

$$\mathbb{E}[[Z(S(L))]] = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial \log B(x) + \boxed{c_1(L, h_L)} \geq 0 .$$

Today : Equidistribution result in semi-classical limit when $(L, h_L) > 0$.

§1 Equidistribution in semi-classical limit

(H) $\begin{cases} \omega = c_1(L, h_L) > 0 & \text{"prequantum condition"} \\ g^{TX} \text{ is complete} \\ \mathrm{Ric}_\omega \geq -C\omega . \end{cases}$

$$(L^p, h_p) := (L^{\otimes p}, h_L^{\otimes p})$$

$p \in \mathbb{N}$

$\rightsquigarrow \quad H^0_{(2)}(X, L^p)$ "quantum state space"

$p \sim 1/\hbar \to +\infty$

$d_p := \dim H^0_{(2)}(X, L^p)$

$p \nearrow \infty \qquad d_p \gg 0$

$\gtrsim C p^n$

$B_p(x, y)$ Bergman kernel

$B_p(x) = B_p(x, x)$

Def : $\underbrace{S(L^p)}_{\substack{\cap \\ H^0(X, L^p)}}$ $p \in \mathbb{N}$ standard Gaussian hol. sections independently defined for $p \in \mathbb{N}$.

"$d_p = \infty$"

(H) $\Rightarrow$ Bergman kernel expansion

Thm 3 (Ma – Marinescu 2007)

Under (H), $\exists\ b_r(x) \in C^\infty(X, \mathbb{R})$, $r = 0, 1, 2, \dots$ $\qquad b_0(x) \equiv 1$

s.t. $\forall\ K \subset\subset X$, $\forall\ k, \ell \in \mathbb{N}$, $\exists\ C_{k,\ell,K} > 0$

s.t.

$$\longrightarrow \left| \frac{1}{p^n} B_p(x) - \sum_{r=0}^{k} b_r(x)\, p^{-r} \right|_{C^\ell(K)} \leq C_{k,\ell,K}\, p^{-k-1}$$

$$\left(B_p(x) \simeq p^n + b_1 p^{n-1} + b_2 p^{n-2} + \cdots \right)$$

Rk: For cpt Kähler mfd

Tian's theorem $\qquad$ peak section method

Then Catlin, Zelditch, Lu ----

Szegö kernel expansion

of Boutet de Monvel – Sjöstrand.

Here, "Analytic Localization"

Bismut – Lebeau, from local index theory

Dai – K. Liu – Ma, Near-diagonal expansion of Bergman kernel

Ma – Marinescu: symplectic case, computation of b_j.

our main reference

Ma – Zhang: G-invariant Bergman kernels

Rk: (H) can be generalized to: (Δ)

g^{TX} complete, $\quad \sqrt{-1} R^L \geq \varepsilon_0 \omega \qquad |\partial \omega| \leq C_0$

$\mathrm{Ric}_\omega \geq -C\omega$

$$b_0(x) = \frac{c_1(L, h_L)^n}{\omega^n} \geq \left(\frac{\varepsilon_0}{2\pi}\right)^n > 0$$

Thm 4 (Equidistribution: Shiffman - Zelditch for cpt Kähler
Drewitz - L. - Marinescu)

Under (H) we have

(1) $$\underbrace{\frac{1}{p}\mathbb{E}\left[[Z(S(L^p))]\right]}_{\downarrow\, p\to+\infty \atop c_1(L,h_L)} = c_1(L,h_L) + \sum_{r\geq 2} p^{-r}\beta_r$$

β_r C^∞ $(1,1)$-form p^{-2}

Equidistribution
$\Downarrow$
$c_1(L,h_L)$

(2) $\forall\ \varphi \in \Omega_0^{n-1,n-1}(X)$, we have

$$\mathbb{P}\left(\lim_{p\to+\infty}\frac{1}{p}\langle[Z(S(L^p))],\varphi\rangle = \langle c_1(L,h_L),\varphi\rangle\right) = 1$$

when X is compact

$$\mathbb{P}\left(\lim_{p\to+\infty}\frac{1}{p}[Z(S(L^p))] = c_1(L,h_L)\right) = 1$$

§ 2. Proof of Thm 4

(1) $$\gamma_{FS}(L^p,h_p) = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\log B_p(x) + p\,c_1(L,h_L)$$

$\|$ Thm 2 (Lecture 1)

$$\mathbb{E}\left[[Z(S(L^p))]\right]$$

$$\frac{1}{p}\mathbb{E}\left[[Z(S(L^p))]\right] = c_1(L,h_L) + \underbrace{\frac{\sqrt{-1}}{2\pi p}\partial\bar\partial\log B_p(x)}$$

$$\frac{1}{p}\log B_p(x) = \frac{1}{p}\log p^n + \frac{1}{p}\log\left(1+\frac{b_1(x)}{p}+\frac{b_2(x)}{p^2}+\cdots\right)$$

$$= \underbrace{\frac{1}{p}\log p^n}_{\partial\bar\partial(\ )=0} + \sum_{r\geq 2}\frac{c_r(x)}{p^r}$$

$$\beta_r(x) = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial c_r(x)$$

(2) Kolmogorov's two series thm : $(Y_p)_p$ random variables in $\mathbb{C}$

$\mathbb{E}[Y_p] = 0$

If $\sum_p \mathbb{E}[|Y_p|^2] < +\infty \Rightarrow \sum_p Y_p$ converges a.s.

$\Rightarrow Y_p \to 0$ a.s.

φ test form

$$Y_p = \left\langle \frac{1}{p}[Z(S(L^p))] - \frac{1}{p}\underbrace{\gamma_{FS}(L^p, h_p)}_{(1,1)\text{-form}}, \varphi \right\rangle$$

$c_1(L, h_L)$

$\mathbb{E}[Y_p] = 0$

Poincaré-Lelong

$$Y_p = \frac{\sqrt{-1}}{2\pi p}\int_X \log\left|\frac{S(L^p)(x)}{\sqrt{B_p(x)}}\right|^2_{h_p} \partial\bar\partial\varphi(x)$$

$$\mathbb{E}[|Y_p|^2] = \frac{1}{4\pi^2 p^2}\int_{\substack{X\times X\\(x,y)}} \mathbb{E}\left[\log\left|\frac{S(L^p)(x)}{\sqrt{B_p(x)}}\right|^2_{h_p}\log\left|\frac{S(L^p)(y)}{\sqrt{B_p(y)}}\right|^2_{h_p}\right] \cdot \underbrace{\partial\bar\partial\varphi(x)\wedge\partial\bar\partial\varphi(y)}_{\text{compact support}}$$

Fubini-Tonelli

Similar in proof of Thm 2

$$\frac{S(L^p)(x)}{\sqrt{B_p(x)}}\Big/ e_L^{\otimes p}(x) \sim N_{\mathbb{C}}(0,1) \qquad \eta \sim N_{\mathbb{C}}(0,1)$$

$\forall\, x, y$

$$\left|\mathbb{E}\left[\log\left|\frac{S(L^p)(x)}{\sqrt{B_p(x)}}\right|^2_{h_p}\log\left|\frac{S(L^p)(y)}{\sqrt{B_p(y)}}\right|^2_{h_p}\right]\right| \leq \mathbb{E}[(\log|\eta|^2)^2] = C$$

$$\mathbb{E}[|Y_p|^2] \leq \frac{C_\varphi}{p_\Delta^2} \Rightarrow Y_p \xrightarrow[a.s]{} 0.$$

when X is cpt $\forall\, s_p \in H^0(X, L^p)$ $s_p \neq 0$

$$\Big|\langle \underbrace{\tfrac{1}{p}[Z(s_p)]}_{\Delta}, \varphi \rangle\Big| \leq |\varphi|_{C^0(X)} \Big|\underbrace{\langle \tfrac{1}{p}[Z(s_p)], \omega^{n-1}\rangle}_{= \int_X \omega^n}\Big|$$

Considering a countable family of $\{\varphi_j\}$ C^0-dense in $\Omega_0^{n-1,n-1}(X)$

$$\mathbb{P}(\text{limit holds with } \varphi_j) = 1$$

$$\Rightarrow \mathbb{P}(\text{limit holds for all } \varphi_j) = 1$$

$$\Rightarrow \mathbb{P}\Big(\tfrac{1}{p}[Z(S(L^p))] \to c_1(L, h_L)\Big) = 1.$$

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Example (SU(2)-poly, Bogomolny-Bohigas-Leboeuf 1996, Shiffman-Zelditch 1999)

quantum chaotic dynamics

$(X, \omega) = (\mathbb{C}\mathbb{P}^1, \omega_{FS})$ $\quad n=1$

$(L, h_L) = (\mathcal{O}(1), h_{FS})$

$H^0(X, L^p) = \mathrm{Span}_{\mathbb{C}}\{1, z, \dots, z^p\}$

$B_p(x) \equiv p+1$

$$S(L^p) = \sum_{j=0}^{p} \eta_j \sqrt{(p+1)\binom{p}{j}}\, z^j$$

SU(2)-poly. elliptic model

$$\mathbb{E}_{SU(2)}\Big[\tfrac{1}{p} Z(S(L^p))\Big] = \omega_{FS} \text{ on } \mathbb{C}\mathbb{P}^1$$

§3 Bergman kernel expansion (Ma-Marinescu)

§3.1 Spectral gap

$$\Box_p\big|_{L^2(X,L^p)} = \bar\partial^{L^p *}\bar\partial^{L^p}$$

$$(H)\quad \begin{cases} \boxed{\omega = c_1(L, h_L)} > 0 \\ Ric_\omega \geq -C\omega \\ g^{TX} \text{ complete} \end{cases}$$

$\Omega^{0,0}$ Hodge Theorem

$$D_p = \sqrt{2}\left(\bar\partial^{L^p} + \bar\partial^{L^p *}\right) \circlearrowright \Omega_0^{0,\bullet}(X, L^p) \subset L^2(X, \Lambda^{0,\bullet}\otimes L^p)$$

$$\Box_p = \frac{1}{2} D_p^2 = \bar\partial^{L^p}\bar\partial^{L^p *} + \bar\partial^{L^p *}\bar\partial^{L^p}$$

$\ker \Box_p = \ker\bar\partial \cap \ker\bar\partial^*$

Kodaira Laplacian

g^{TX} complete: has a unique self-adjoint extension, Gaffney extension.

$$H^0_{(2)}(X, L^p) = \ker\left(\bar\partial^{L^p}\big|_{L^2(X,L^p)}\right) = \ker\left(\Box_p\big|_{L^2(X,L^p)}\right)$$

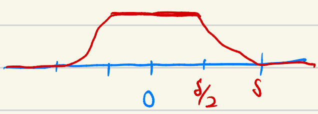
Prop (Spectral gap) $\exists\ C_1, C_2 > 0$

$$\mathrm{Spec}(\Box_p) \subset \{0\} \underbrace{\sqcup}_{\text{"gap"}} [C_1 p - C_2, +\infty)$$

$\delta > 0$

$$h(s) = \begin{cases} 1 & |s| \leq \delta/2 \\ 0 & |s| \geq \delta \end{cases}$$

even



$$H(a) = \left(\int_{-\infty}^{\infty} h(s)\,ds\right)^{-1} \int_{-\infty}^{\infty} e^{\sqrt{-1}sa} h(s)\,ds$$

Schwartz function, $a \in \mathbb{R}$

$$\phi_p(a) = 1_{[\sqrt{C_1 p - C_2}, +\infty)}\left(\frac{|a|}{\sqrt{2}}\right) H(a)$$

速降

$$H(D_p) - B_p = \phi_p(D_p) = O(p^{-\infty})$$
$$(\le C_\ell p^{-\ell} \quad \forall \ell)$$

Taking integral kernels

$$H(D_p)(x,x') - B_p(x,x') = O(p^{-\infty})$$
$$B_p(x,x') = \underbrace{H(D_p)(x,x')}_{\text{main term}} + O(p^{-\infty}).$$

$\Rightarrow$ (1) $H(D_p)(x,x')$ only depends on $D_p|_{B^X(x,\delta)}$

(2) if $d(x,x') > \delta$ (↑ distance), $H(D_p)(x,x') = 0$.

Cor 1 $B_p(x,x') = O(p^{-\infty})$ for all $d(x,x') > \delta$, $x, x' \in$ cpt.

§3.2 Localization Principle

(M_1, ω_1), (M_2, ω_2) complete Kähler of dim $= n$

$(L_1, h_{L_1}) \to M_1$ $\quad c_1(L_j, h_{L_j}) = \omega_j$

$(L_2, h_{L_2}) \to M_2$ $\quad$ & $\mathrm{Ric}_{\omega_j} \ge -C\omega_j$

Assume $\exists\, U_j \subseteq M_j$ $\quad \Phi : (L_1|_{U_1}, h_{L_1}) \xrightarrow{\sim} (L_2|_{U_2}, h_{L_2})$ iso

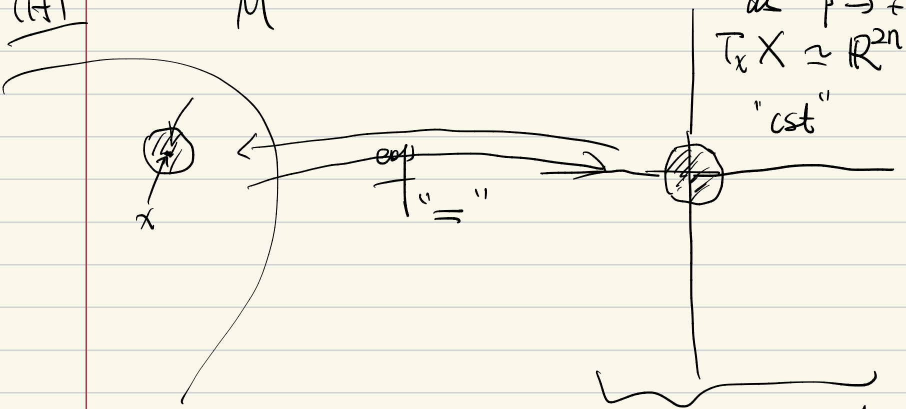
$$\begin{array}{ccc} (L_1|_{U_1}, h_{L_1}) & \xrightarrow{\sim} & (L_2|_{U_2}, h_{L_2}) \\ \downarrow & & \downarrow \\ \Phi \quad U_1 & \underset{\text{bih}}{\xrightarrow{\sim}} & U_2 \end{array}$$

Then $\forall\, k, m \in \mathbb{N}$, $\forall\, K \subset\subset W_1$

$$B_{1,p} - B_{2,p} \circ (\bar{\Phi}, \Phi) = O(p^{-k})$$

in $C^m(K \times K)$ as $p \to +\infty$.

(H)



$T_xX \simeq \mathbb{R}^{2n} \simeq \mathbb{C}^n$

"cst"

Bargmann–Fock model with small perturbation near origin

§3.3 Near-diagonal expansion

$L = \mathbb{C} \to \mathbb{C}^n$ $\quad dV$

$\mathcal{L}^2(\mathbb{C}^n)$

$$\bar{\partial}_{BF} = \sum_j d\bar{z}_j \wedge \left(2 \frac{\partial}{\partial \bar{z}_j} + \pi z_j\right)$$

$$\mathcal{H}^0_{(2)}(\mathbb{C}^n) = \ker(\bar{\partial}_{BF})$$

$$\mathcal{B} : \mathcal{L}^2(\mathbb{C}) \xrightarrow[\Delta]{\perp} \mathcal{H}^0_{(2)}(\mathbb{C}^n)$$

ONB of $\mathcal{H}^0_{(2)}(\mathbb{C}^n)$ $\quad \left(\frac{\pi^{|\alpha|}}{\alpha!}\right)^{1/2} z^\alpha \exp\left(-\frac{\pi}{2}|z|^2\right)$ $\quad \forall\, \alpha \in \mathbb{N}^n$

$$\mathscr{B}(Z, Z') = \underline{\exp\left(-\frac{\pi}{2}\left(|z_j|^2 + |z_j'|^2 - 2 z_j \bar{z}_j'\right)\right)}$$

$$\begin{cases} Z \in \mathbb{R}^{2n} \simeq \mathbb{C}^n \\ z_j = Z_{2j-1} + \sqrt{-1}\, Z_{2j} \end{cases}$$

$$D_p^2 \quad \widetilde{\ \ } \quad \frac{1}{\sqrt{p}} \quad \underbrace{\sum_j \left(-2\frac{\partial}{\partial z_j} + \pi \bar{z}_j\right)\left(2\frac{\partial}{\partial \bar{z}_j} + \pi z_j\right)} + \sum_{r \geq 1} O_r \, p^{-r/2}$$

<u>Thm 5</u>: $x \in U \subset\subset X$, $Z, Z' \in T_x X$, $|Z|_{g^{TX}}, |Z'|_{g^{TX}} \leq \delta/2$

$$\left| \frac{1}{p^n} B_p(\exp_x(Z), \exp_x(Z')) - \sum_{r=0}^{N} p^{-r/2} J_r(\sqrt{p} Z, \sqrt{p} Z') \mathscr{B}(\sqrt{p} Z, \sqrt{p} Z') \kappa^{-1/2}(Z) \kappa^{-1/2}(Z') \right|_{C^\ell_{Z,Z'}}$$

$$\leq C p^{-\frac{N-\ell+1}{2}} \left(1 + \sqrt{p}(|Z| + |Z'|)\right)^{M(N,\ell)} \cdot \exp(-C\sqrt{p}\,|Z - Z'|) + O(p^{-\infty})$$

— J_r poly $\deg_{Z,Z'} \leq 3r$

$J_0 \equiv 1$ $\qquad \kappa = \dfrac{dv^X}{dv^{T_xX}}$ $\qquad \kappa(0) = 1$, C^∞

— <u>r odd</u> $\quad J_r$ odd function

$J_r(0,0) = 0$

— $Z = Z' = 0$ $\qquad B(0,0) = 1$ $\Rightarrow$ <u>Thm 3</u>

$\Rightarrow \quad B_p(x) = p^n + b_1(x) p^{n-1} \cdots$

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